

The Cooperative Cleaners Problem in Stochastic Dynamic Environments

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In this paper we study the strengths and limitations of collaborative teams of simple agents. In particular, we discuss the efficient use of “ant robots” for covering a connected region on the \mathbf{Z}^2 grid, whose area is unknown in advance and which expands stochastically. Specifically, we discuss the problem where an initial connected region of S_0 boundary tiles expand outward with probability p at every time step. On this grid region a group of k limited and simple agents operate, in order to clean the unmapped and dynamically expanding region. A preliminary version of this problem was discussed in [1, 2], involving a deterministic expansion of a region in the grid. In this work we extend the model and examine cases where the spread of the region is done stochastically, where each tile has some probability p to expand, at every time step. For this extended model we obtain an analytic probabilistic lower bounds for the minimal number of agents and minimal time required to enable a collaborative coverage of the expanding region, regardless of the algorithm used and the robots hardware and software specifications. In addition, we present an impossibility result, for a variety of regions that would be impossible to completely clean, regardless of the algorithm used. Finally, we validate the analytic bounds using extensive empirical computer simulation results.

I. INTRODUCTION

In this work we discuss in this work is the *Cooperative Cleaners* problem — a problem assuming a regular grid of connected rooms (pixels), some of which are ‘dirty’ and the ‘dirty’ pixels forming a connected region of the grid. On this dirty grid region several agents operate, each having the ability to ‘clean’ the place (the ‘room’, ‘tile’, ‘pixel’ or ‘square’) it is located in. We examine problems in which the agents work in stochastic dynamic environments — where probabilistic changes in the environment may take place and are independent of, and certainly not caused by, the agents’ activity. In the spirit of [3] we consider simple robots with only a bounded amount of memory (i.e. *finite-state-machines*).

The static variant of this problem was introduced in [4], where a cleaning protocol ensuring that a decentralized group of agents will jointly clean any given (and a-priori unknown) dirty region. The protocol’s performance, in terms of cleaning time was fully analyzed and also demonstrated experimentally.

A dynamic generalization of the problem was later presented in [2], in which a deterministic expansion of the environment is assumed, simulating the spreading of a *contamination* (or a spreading “danger” zone or *fire*). Once again, the goal of the agents is to clean the spreading contamination in as efficiently as possible.

In this work we modify the ‘dirty’ region expansion model and add stochastic features to the spreadings of the region’s cells. We formally define and analyze the *Cooperative Cleaning* problem, this time under this stochastic generalization.

We focus on a variant of the *Cooperative Cleaning* problem, where the tiles have some probability to be contaminated by their neighbors’ contamination. This version of the problem has applications in the “real” world and in the computer network environments as well. For instance, one of the applications can be a distributed anti-virus software trying to overcome an epidemic malicious software attacking a network of computers. In this case, each infected computer has some probability to infect computers connected to it.

A more general paradigm of the cleaning problem is when the transformation of the contaminated area from one state to another is described in the form of some pre-defined function. For instance, following the previous example, we can say that the sub-network affected by the virus, is spreading by a certain rule. We can say that a computer will be infected by a virus with a certain probability, which depends on the number of the neighboring computers already infected. By defining rules for the contamination’s spreading and cleaning, we can think of to this problem as a kind of Conway’s “*Game of Life*”, where each cell in the game’s grid spreads its “seed” to neighboring cells (or alternatively, “dies”) according to some basic rules.

While the problem posed in [2], as well as the analysis methods used and the correctness proofs, were all deterministic, it is interesting to examine the stochastic variant of such algorithms. In this work we analyze and derive a lower bound on the expected cleaning time for k agents running a cleaning protocol under a model, where every tile

in the neighborhood of the affected region may become contaminated at every time step with some probability, the contamination coming from its dirty neighbors.

II. ORGANIZATION

The main contributions of this paper are a bound on the contaminated region's size and a bound on the cleaning time which is presented in Section V. We also present a method which bounds the cleaning time for a given desired probability. We then provide an impossibility result for the problem raised in Section VI. The rest of paper is organized as followed: In Section III we survey some of the related works. In Section IV we formalize the problem, giving the basic definitions needed for the later analysis, in Section VII we present some of the experimental results and compare them to the analytic bounds and concluding in Section VIII.

III. RELATED WORK

Significant research effort is invested in the design and simulation of multi-agent robotics and intelligent swarm systems (see e.g. [5–10]).

In general, most of the techniques used for the distributed coverage of some region are based on some sort of cellular decomposition. For example, in [11] the area to be covered is divided between the agents based on their relative locations. In [12] a different decomposition method is being used, which is analytically shown to guarantee a complete coverage of the area. [13] discusses two methods for cooperative coverage (one probabilistic and the other based on an exact cellular decomposition).

While some existing works concerning distributed (and decentralized) coverage present analytic proofs for the ability of the system to complete the task (for example, in [12–14]), most of them lack analytic bounds for the coverage time (and often extensive amounts of empirical results on this are made available by extensive simulations). Although a proof for the coverage completion is an essential element in the design of a multi-agent system, analytic indicators for its efficiency are in our opinion of great importance. We provide such results, as bounds for the cleaning time of the agents, in Section V.

An interesting work to mention in this context is that of Koenig and his collaborators [15, 16], where a swarm of ant-like robots is used for repeatedly covering an unknown area, using a real time search method called *node counting*. By using this method, the robots are shown to be able to efficiently perform a coverage mission, and analytic bounds for the coverage time are discussed.

Another work discussing a decentralized coverage of terrains is presented in [17]. This work examines domains with non-uniform traversability. Completion times are given for the proposed algorithm, which is a generalization of the forest search algorithm. In this work, though, the region to be searched is assumed to be known in advance - a crucial assumption for the search algorithm, which relies on a cell-decomposition procedure.

Vertex-Ant-Walk, a variation of the node counting algorithm is presented in [8] and is shown to achieve a coverage time of $O(n\delta_G)$, where δ_G is the graph's diameter, which is based on a previous work in which a cover time of $O(n^2\delta_G)$ was demonstrated [18]. Another work called *Exploration as Graph Construction*, provides a coverage of degree bounded graphs in $O(n^2)$ time, is described in [19]. Here a group of ant robots with a limited capability explores an unknown graph using special “markers”.

Similar works concerning multi agents systems may be found in [11–14, 20–25]).

The *Cooperative Cleaning* problem is also strongly related to the problem of distributed search after mobile and evading target(s) [26–29] or the problems discussed under the names of “Cops and Robbers” or “Lions and Men” pursuits [30–35].

IV. DEFINITIONS

In our work we will use some of the basic notations and definitions, that were used in [2], which we shell briefly review. As in the above mentioned previous works on this problem, we shall assume that the time is discrete.

Definition 1. Let an undirected graph $G(V, E)$ describe the two dimensional integer grid \mathbf{Z}^2 , whose vertices (or “tiles”) have a binary property called “contamination”. Let $cont_t(v)$ denote the contamination state of the tile v at time t , taking either the value “on” (for “dirty” or “contaminated”) or “off” (for “clean”).

For two vertices $v, u \in V$, the edge (v, u) may belong to E at time t only if both of the following hold : a) v and u are 4-Neighbors in G . b) $cont_t(v) = cont_t(u) = on$. This however is a necessary but not a sufficient condition as we elaborate below.

The edges of E represent the connectivity of the contaminated region. At $t = 0$ all the contaminated tiles are connected, namely :

$$(v, u) \in E_0 \iff (v, u \text{ are 4-Neighbors in } G) \wedge (cont_0(v) = cont_0(u) = on)$$

Edges may be added to E only as a result of a contamination spread and can be removed only while contaminated tiles are cleaned by the agents.

Definition 2. Let $F_t(V_{F_t}, E_t)$ be the contaminated sub-graph of G at time t , i.e. :

$$V_{F_t} = \{v \in G \mid cont_t(v) = on\}$$

We assume that F_0 is a single simply-connected component (the actions of the agents will be so designed that this property will be preserved).

Definition 3. Let ∂F denote the boundary of F . A tile is on the boundary if and only if at least one of its 8-Neighbors is not in F , meaning :

$$\partial F = \{v \mid v \in F \wedge 8\text{-Neighbors}(v) \cap (G \setminus F) \neq \emptyset\}$$

Definition 4. Let S_t denote the size of the dirty region F at time t , namely the number of grid points (or tiles) in F_t .

Let a group of k agents that can move on the grid G (moving from a tile to its neighbor in one time step) be placed at time t_0 on F_0 , at some point $p_0 \in V_{F_t}$.

Definition 5. Let us denote by ΔF_t the potential boundary, which is the maximal number of tiles which might be added to F_t by spreading all the tiles of ∂F_t .

$$\Delta F_t \equiv \{v : \exists u \in \partial F_t \text{ and } v \in 4\text{-Neighbors}(u) \text{ and } v \notin F_t\}$$

As we are interested in the stochastic generalization of the dynamic cooperative cleaners model, we will assume that each tile in ΔF_t might be contaminated with some probability p . In the model we will analyze later, we assume that the status variables of the tiles of ΔF_t are independent from one another, and between time steps.

Definition 6. Let us denote by $\Phi_n(v)$ the surrounding neighborhood of a tile v , as the set of all the reachable tiles u from v within n steps on the grid (namely, the “digital sphere” or radius n around v). In this work we assume 4-connectivity among the region cells — namely, two tiles are considered as neighbors within one step iff the Manhattan distance between them is exactly 1.

The spreading policy, $\Xi(v, \phi, t)$, controls the contamination status of v at time $t + 1$, as a function of the contamination status of its neighbors in its n -th digital sphere, at time t . Notice the $\Xi(v, \phi, t)$ can be also non deterministic.

Definition 7. Let us denote by $\Xi(v, \phi, t)$ the spreading policy of v as follows:

$$\Xi(v, \phi, t) : (V, \{On, Off\}^\alpha, \mathbb{N}) \rightarrow \{On, Off\}$$

Where $\alpha \equiv |\Phi_n(v)|$, V denotes the vertices of the grid, $\{On, Off\}^\alpha$ denotes the contamination status of the members of $\Phi_n(v)$, at time t (for $t \in \mathbb{N}$).

A basic example of using the previous definition of $\Xi()$ is the case of the *deterministic model*, where at every d time-steps the contamination spreads from all tiles in ∂F_t to all the tiles in ΔF_t . This model can be defined using the Ξ function, as follows:

For every tile v we first define $\Phi_n(v)$ where n equals to 1 and assuming 4-connectivity. Then $\Xi(v, \phi, t)$, for any time-step t will be defined as follows:

$$\Xi(v, \phi, t) = \begin{cases} On & \text{if } t \bmod d = 0 \text{ and } v \in \Delta F_t \\ Off & \text{Otherwise} \end{cases}$$

Notice that due to the fact that we are assuming that v is in ΔF_t , its *surrounding neighborhood* contains at least one tile with contamination status of On .

An interesting particular case of the general $\Xi()$ function is the *simple uniform probabilistic spread*. In this scenario, a tile in $V \in \Delta F_t$ becomes contaminated with some predefined probability p , if and only if at least one of its n -th neighbors are contaminated at time step t . Using the $\Xi()$ function and the probability p , this can be formalized as follows :

For every tile v we first define $\Phi_n(v)$ where n equals to 1 and assuming 4-connectivity.

$$\Xi(v, \phi, t) = \begin{cases} \text{On} & \text{with probability of } p \text{ if } v \in \Delta F_t \\ \text{Off} & \text{Otherwise} \end{cases}$$

This model can naturally also be defined simply as :

$$\forall t \in \mathbb{N}, \forall v \in \Delta F_t, \text{Prob}(\text{cont}_{t+1}(v) = \text{On}) = p$$

In our work we will focus on this model, while deriving the analytic bounds for the cleaning time.

V. LOWER BOUND

A. Direct Bound

In this section we form a lower bound on the cleaning time of any cleaning protocol preformed by k agents. We start by setting a bound on the contaminated region's size at each time step, S_t . As we are interested in minimizing the cleaning time we should also minimize the contaminated region's area. Therefore we are interested in the minimal size of it, which achieved when the region's shape is sphere [2]. In our model each tile in the *potential boundary*, ΔF_t , has the same probability p to be contaminated in the next time step. The whole probabilistic process in each time step is *Binomial Distributed*, under the assumption that the spreading of each tile at any time step is independent from the spreadings of other tiles or from the spreadings of the same tile at different time steps.

As we are interested in the lower bound of the contaminated region's size we will assume that the expected number of newly added tiles to the contaminated region is minimal, which occurs when the region's shape forms a digital sphere (as presented in [36]). Then we can compute the expectation of this process for a specific time step t . Therefore, the size of the *potential boundary* is $\Delta F_t = 2\sqrt{2} \cdot S_t - 1$ as shown in [36, 37].

Definition 8. Let us denote by X_t the random variable of the actual number of added tiles to the contaminated region at time step t .

Assuming the independence of tiles' contamination spreadings and given S_t , X_t is *Binomial Distributed*, $X_t|S_t \sim B(\Delta F_t, p)$, where each tile in the *potential boundary* has the same probability p to be contaminated. Therefore, we can say that the expectation of X_t given S_t is $\mu = E(X_t|S_t) = p \cdot \Delta F_t$.

Notice that occasionally the number of new tiles added to the contaminated region may be below μ . As we are interested in a lower bound, we should take some $\mu' < \mu$ such that: $\Pr[X_t < \mu'|S_t] < \epsilon$, meaning that the probability that the number of the newly added tiles to the contaminated region is smaller than μ' is extremely small (tends to 0).

In order to bound X_t by some μ' we shall use the *Chernoff Bound*, where δ is the desired distance from the expectation, as follows :

$$\Pr[X_t < (1 - \delta)\mu|S_t] < e^{-\frac{\delta^2\mu}{2}}$$

Definition 9. Let us denote by q_t the probability that at time step t , the size of the added tiles to the contaminated region is not lower than $\mu' = (1 - \delta)\mu$ and it can be written as follows :

$$q_t = (1 - \Pr[X_t < (1 - \delta)\mu|S_t])$$

Theorem 1. Using any cleaning protocol, the area of the contaminated region at time step t can be recursively lower bounded, as follows :

$$\Pr[S_{t+1} \geq S_t - k + \left\lfloor 2 \cdot (1 - \delta) p \cdot \sqrt{2 \cdot (S_t - k) - 1} \right\rfloor | S_t] \geq q_t$$

Proof. Notice that a lower bound for the contaminated region's size can be obtained by assuming that the agents are working with maximal efficiency, meaning that each time step every agent cleans exactly one tile.

In each step, the agents clean another portion of k tiles, but the remaining contaminated tiles spread their contamination to their 4-*Neighbors* and cause new tiles to be contaminated.

Lets us denote by the random variable S_{t+1} the number of contaminated tiles in the next time step. Using Definition 8 we can express S_{t+1} as follows :

$$S_{t+1} = S_t - k + X_t$$

Lets first bound the number of the added tiles using the *Chernoff Bound* . As X_t given S_t is *Binomial Distributed*, $X_t|S_t \sim B(\Delta F_t, p)$ and $\mu = E(X_t|S_t) = p \cdot \Delta F_t$. Using *Chernoff Bound* we know that:

$$Pr[X_t < (1 - \delta)\mu|S_t] < e^{-\frac{\delta^2\mu}{2}} \Rightarrow Pr[X_t < (1 - \delta)p \cdot \Delta F_t|S_t] < e^{-\frac{\delta^2 \cdot p \cdot \Delta F_t}{2}}$$

Assigning $X_t = S_{t+1} - S_t + k$ from former definition of S_{t+1} , we get:

$$Pr[S_{t+1} - S_t + k < (1 - \delta)p \cdot \Delta F_t|S_t] < e^{-\frac{\delta^2 \cdot p \cdot \Delta F_t}{2}}$$

As we are interested in the *minimal* number of tiles which can become contaminated at this stage. The minimal number of 4-*Neighbors* of any number of tiles is achieved when the tiles are organized in the shape of a “digital sphere” (see [36, 37]) - i.e. the *potential boundary* is $\Delta F_t = 2\sqrt{2 \cdot (S_t - k) - 1}$. Assigning ΔF_t value:

$$Pr[S_{t+1} < S_t - k + (1 - \delta)p \cdot 2\sqrt{2 \cdot (S_t - k) - 1}|S_t] < e^{-\frac{\delta^2 \cdot p \cdot 2\sqrt{2 \cdot (S_t - k) - 1}}{2}}$$

As we are interested in the complementary event and using Definition 9

$$Pr[S_{t+1} \geq S_t - k + (1 - \delta)p \cdot 2\sqrt{2 \cdot (S_t - k) - 1}|S_t] \geq 1 - e^{-\frac{\delta^2 \cdot p \cdot 2\sqrt{2 \cdot (S_t - k) - 1}}{2}} = q_t \quad (1)$$

As the number of tiles must be an integer value, we use $\lfloor (1 - \delta) \cdot p \cdot 2\sqrt{2 \cdot S_t - 1} \rfloor$ to be on the safe side. Using inequality 1 we get :

$$Pr[S_{t+1} \geq S_t - k + \lfloor 2 \cdot (1 - \delta) p \cdot \sqrt{2 \cdot (S_t - k) - 1} \rfloor | S_t] \geq q_t$$

□

Notice that as illustrated in Figure 1(a), which demonstrates the bound presented in Theorem 1, as δ decreases the produced bound for the stochastic model is closer to the bound of the deterministic model, for $d = \frac{1}{p}$.

Definition 10. Let us denote by Q_t the bound probability that the contaminated region's size at time step t will be [at least] S_t . Q_t can be expressed as follows :

$$Q_t = \prod_{i=0}^t q_i$$

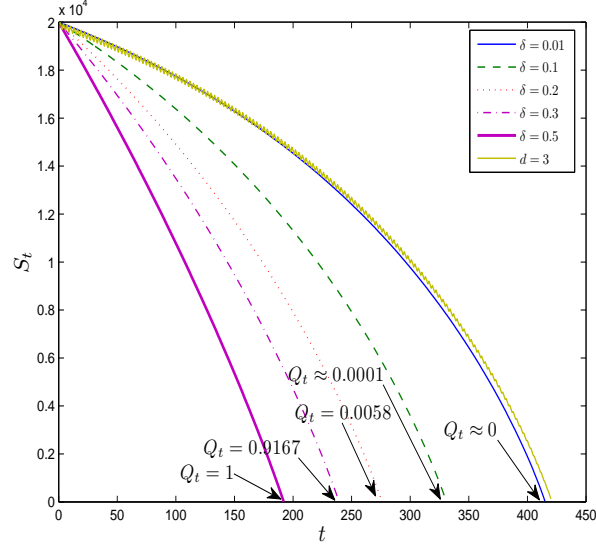
Notice that for bounding the area of the region at time step t using Theorem 1 and Definition 10, the bound, which will be achieved, will be in probability Q_t . We want to have Q_t sufficiently high.

We shall assume, for the sake of analysis, that the dynamic value of the area, S_t , is always kept not less than some $\hat{S} < S_0 - k + \lfloor 2 \cdot (1 - \delta) p \cdot \sqrt{2 \cdot (S_0 - k) - 1} \rfloor$ (as we want S_1 to be bigger or equal to \hat{S}). Then the next Lemma holds :

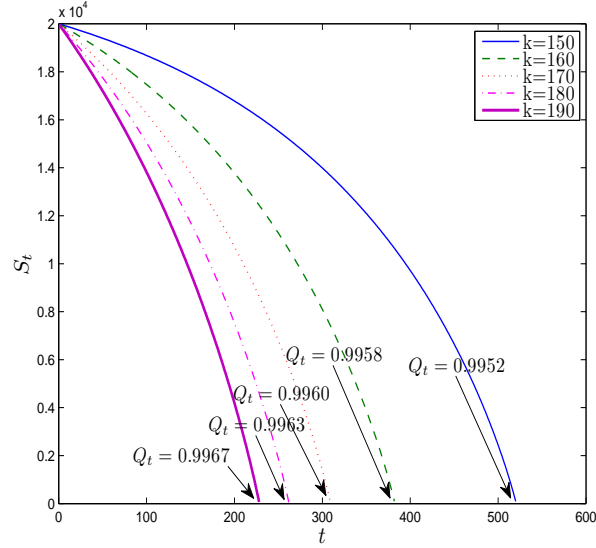
Lemma 1. For any $T \geq 1$, if for all $1 \leq t \leq T$ the contaminated region's size S_t is always kept not less than some $\hat{S} < S_0 - k + \lfloor 2 \cdot (1 - \delta) p \cdot \sqrt{2 \cdot (S_0 - k) - 1} \rfloor$ then :

$$Q_T \geq \hat{Q}_T = \hat{q}^T = \left(1 - e^{-\delta^2 \cdot p \cdot 2\sqrt{2 \cdot (\hat{S} - k) - 1}}\right)^T$$

Proof. We will prove this Lemma by induction on T.



(a) A lower bound for the contaminated region S_t , the area at time t , for various values of δ



(b) A lower bound for the contaminated region S_t , the area at time t , for various number of agents k

FIG. 1: An illustration of the bound presented in Theorem 1. In (a) we can see the deterministic model (the Zig-Zag line) with spread each 3 time steps ($d = 3$) compare to the stochastic model with $p = 1/3$ and $\delta \in [0.01, 0.1, 0.2, 0.3, 0.5]$ where both models have $k = 150$ and start with $S_0 = 20000$. In (b) we have the lower bound for $S_0 = 20000$, $p = 0.5$ and $\delta = 0.3$ for different number of agents $k \in [150, 160, 170, 180, 190]$.

- *Base step:* For $T = 1$, we can write the probabilities Q_1 and \hat{Q}_1 as follows :

$$Q_1 = q_1 \geq 1 - e^{-\delta^2 \cdot p \cdot 2\sqrt{2 \cdot (S_1 - k) - 1}}$$

$$\hat{Q}_1 = \hat{q} \geq 1 - e^{-\delta^2 \cdot p \cdot 2\sqrt{2 \cdot (\hat{S} - k) - 1}}$$

As we assume that $S_t \geq \hat{S}$ than it is not hard to see that $Q_1 \geq \hat{Q}_1$.

- *Induction hypothesis:* We assume that for all $T \leq T'$ holds $Q_T \geq \hat{Q}_T$.
- *Induction step:* We will prove that for $T = T' + 1$ holds $Q_T \geq \hat{Q}_T$. From definition 10 we can write the probability Q_T as follows :

$$Q_T = Q_{T'+1} = Q_{T'} \cdot q_{T'+1}$$

By the induction hypothesis we know that for all $T \leq T'$ holds $Q_T \geq \hat{Q}_T$ than we can rewrite $Q_{T'+1}$ and also writing \hat{Q}_T for $T = T' + 1$ we get that :

$$Q_T = Q_{T'+1} \geq \hat{Q}_{T'} \cdot q_{T'+1}$$

$$\hat{Q}_T = \hat{Q}_{T'+1} = \hat{Q}_{T'} \cdot \hat{q}$$

As we want to compare these probabilities and to prove that $Q_T \geq \hat{Q}_T$ all we need to show is that $q_{T'+1} \geq \hat{q}$. As we assume that for all $1 \leq t \leq T' + 1$ holds that $S_t \geq \hat{S}$ and particularly for $t = T' + 1$, than it is not hard to see that $q_{T'+1} \geq \hat{q}$ and therefore $Q_T \geq \hat{Q}_T$.

□

Theorem 2. For any contaminated region of size S_0 , using any cleaning protocol, the probability that $S_{\hat{\tau}_\delta}$, the contaminated area at time step $t = \hat{\tau}_\delta$, is greater or equal to some $\hat{S} < S_0 - k + \left\lfloor 2 \cdot (1 - \delta) p \cdot \sqrt{2 \cdot (S_0 - k) - 1} \right\rfloor$ can be lower bounded, as follows :

$$Pr \left[S_{\hat{\tau}_\delta} \geq \hat{S} \right] \geq \left(1 - e^{-\delta^2 \cdot p \cdot 2 \sqrt{2 \cdot (\hat{S} - k) - 1}} \right)^{\hat{\tau}_\delta}$$

where :

$$\hat{\tau}_\delta \triangleq \frac{\sqrt{\varpi \cdot \left(\hat{S} - k - \frac{1}{2} \right)} - \sqrt{\varpi \cdot \left(S_0 - k - \frac{1}{2} \right)} + \ln \left(\frac{\sqrt{\varpi \cdot \left(\hat{S} - k - \frac{1}{2} \right)} - \frac{k}{2}}{\sqrt{\varpi \cdot \left(S_0 - k - \frac{1}{2} \right)} - \frac{k}{2}} \right)^{\frac{k}{2}}}{\varpi}$$

and

$$\varpi \triangleq 2(1 - \delta)^2 \cdot p^2$$

Proof. Observe that by denoting $y_t \triangleq S_t$ Theorem 1 can be written as :

$$y_{t+1} - y_t \geq \left\lfloor 2 \cdot (1 - \delta) \cdot p \sqrt{2 \cdot (y_t - k) - 1} \right\rfloor - k$$

Searching for the minimal area we can look at the equation :

$$y_{t+1} - y_t = \left\lfloor 2 \cdot (1 - \delta) \cdot p \sqrt{2 \cdot (y_t - k) - 1} \right\rfloor - k$$

By dividing both sides by $\Delta t = 1$ we obtain :

$$y_{t+1} - y_t \triangleq y' = \left\lfloor \sqrt{(1 - \delta)^2 \cdot p^2 \cdot 8 \left[y - \left(k + \frac{1}{2} \right) \right]} \right\rfloor - k \quad (2)$$

Notice that the values of y' , the derivative of the change in the region's size, might be positive (stating an increase in the area), negative (stating a decrease in the area), or complex numbers (stating that the area is smaller than k , and will therefore be cleaned before the next time step).

Let us denote $x^2 \triangleq (1 - \delta)^2 \cdot p^2 \cdot 8 \left[y - \left(k + \frac{1}{2} \right) \right]$. After calculating the derivative of both sides of this expression we see that :

$$2x \cdot x' = (1 - \delta)^2 \cdot p^2 \cdot 8y'$$

and after using the definition of y' of Equation 2 we see that :

$$\begin{aligned} 2x \cdot \frac{dx}{dt} = 2x \cdot x' &= (1-\delta)^2 \cdot p^2 \cdot 8 \left(\left\lfloor \sqrt{(1-\delta)^2 \cdot p^2 \cdot 8 \left[y - \left(k + \frac{1}{2} \right) \right]} \right\rfloor - k \right) \\ &\leq (1-\delta)^2 \cdot p^2 \cdot 8 (x - k) \end{aligned} \quad (3)$$

From Equation 3 a definition of dt can be extracted :

$$\begin{aligned} dt &\geq \frac{1}{8 \cdot (1-\delta)^2 \cdot p^2} \cdot \frac{2x}{x-k} dx \\ &\geq \frac{1}{4(1-\delta)^2 \cdot p^2} \cdot \frac{x-k+k}{x-k} dx \geq \frac{1}{4(1-\delta)^2 \cdot p^2} \left(1 + \frac{k}{x-k} \right) dx \end{aligned}$$

The value of x can be achieved by integrating the previous expression as follows (notice that we are interested in the equality of the two expressions) :

$$\int_{t_0}^t dt = \int_{x_0}^x \frac{1}{4(1-\delta)^2 \cdot p^2} \left(1 + \frac{k}{x-k} \right) dx$$

After the integration we can see that :

$$i \Big|_{t_0}^t = \frac{1}{4(1-\delta)^2 \cdot p^2} (x + k \ln(x-k)) \Big|_{x_0}^x$$

and after assigning $t_0 = 0$:

$$4(1-\delta)^2 \cdot p^2 \cdot t = x - x_0 + k \ln \frac{x-k}{x_0-k}$$

Returning back to y and using ϖ definition we get :

$$\varpi \cdot t = \sqrt{\varpi \left(y - k - \frac{1}{2} \right)} - \sqrt{\varpi \left(y_0 - k - \frac{1}{2} \right)} + \ln \left(\frac{\sqrt{\varpi \left(y - k - \frac{1}{2} \right)} - \frac{k}{2}}{\sqrt{\varpi \left(y_0 - k - \frac{1}{2} \right)} - \frac{k}{2}} \right)^{\frac{k}{2}}$$

Returning to the original size variable S_t , we see that :

$$\varpi \cdot t = \sqrt{\varpi \left(S_t - k - \frac{1}{2} \right)} - \sqrt{\varpi \left(S_0 - k - \frac{1}{2} \right)} + \ln \left(\frac{\sqrt{\varpi \left(S_t - k - \frac{1}{2} \right)} - \frac{k}{2}}{\sqrt{\varpi \left(S_0 - k - \frac{1}{2} \right)} - \frac{k}{2}} \right)^{\frac{k}{2}} \quad (4)$$

Defining that $\hat{\tau}_\delta = t$ and combining EQ. 4 with Lemma 1 knowing that $S_{t'} \geq \hat{S}$ for all $1 \leq t' \leq \hat{\tau}_\delta$ we get the following inequality:

$$Pr \left[S_{\hat{\tau}_\delta} \geq \hat{S} \right] \geq \left(1 - e^{-\delta^2 \cdot p \cdot 2 \sqrt{2 \cdot (\hat{S} - k) - 1}} \right)^{\hat{\tau}_\delta}$$

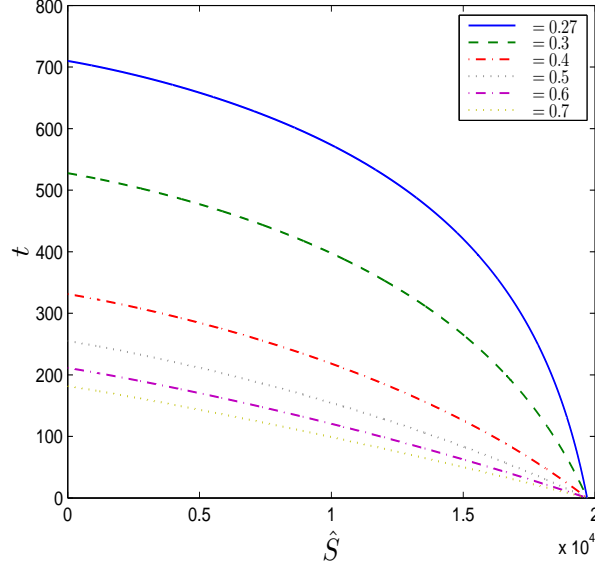
where :

$$\hat{\tau}_\delta \triangleq \frac{\sqrt{\varpi \cdot \left(\hat{S} - k - \frac{1}{2} \right)} - \sqrt{\varpi \cdot \left(S_0 - k - \frac{1}{2} \right)} + \ln \left(\frac{\sqrt{\varpi \cdot \left(\hat{S} - k - \frac{1}{2} \right)} - \frac{k}{2}}{\sqrt{\varpi \cdot \left(S_0 - k - \frac{1}{2} \right)} - \frac{k}{2}} \right)^{\frac{k}{2}}}{\varpi}$$

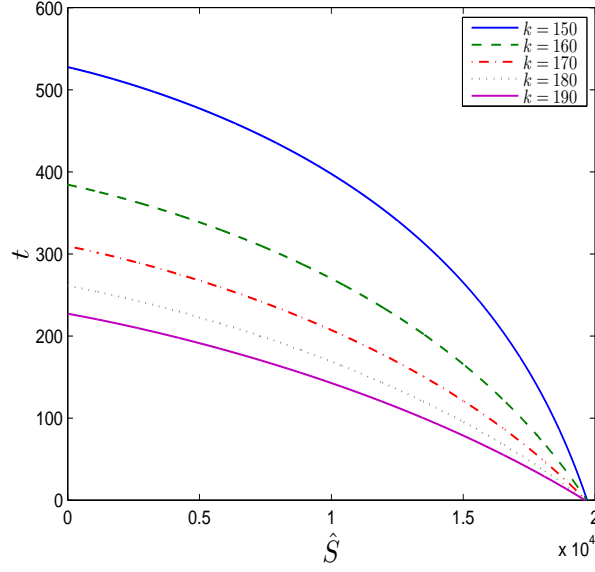
and

$$\varpi \triangleq 2(1-\delta)^2 \cdot p^2$$

□



(a) The bound on t for various values of δ as a function of \hat{S}

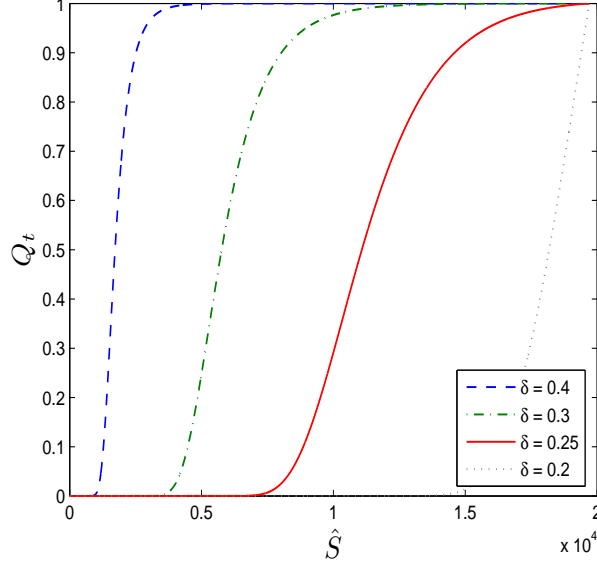


(b) The bound on t for various number of agents k as a function of \hat{S}

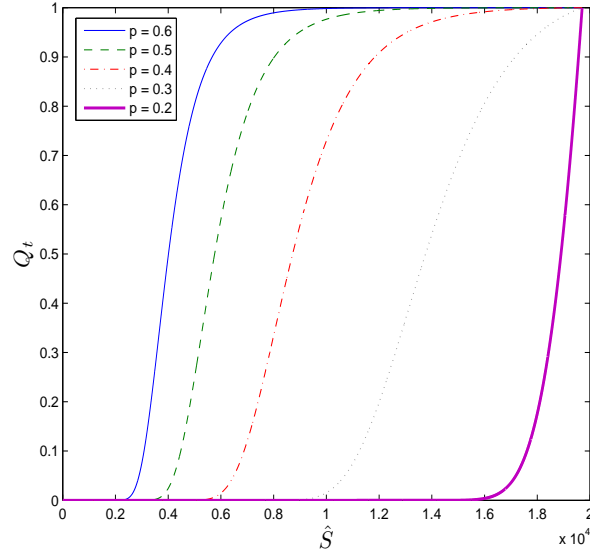
FIG. 2: An illustration of the bound presented in Theorem 2 of the cleaning time t in order to reach \hat{S} . In (a) we can see the bound on the cleaning time where $p = 0.5$ and $\delta \in [0.27, 0.3, 0.4, 0.5, 0.6, 0.7]$ where the cleaning done by $k = 150$ agents and starting with $S_0 = 20000$. In (b) we have the lower bound on the cleaning time for $S_0 = 20000$, $p = 0.5$ and $\delta = 0.3$ for different number of agents $k \in [150, 160, 170, 180, 190]$.

In Theorem 2 we can guarantee with high probability of \hat{Q}_{τ_δ} that the contamination region's size will not be lower than \hat{S} - namely for any time step $t > \tau_\delta$ the probability $Pr[S_t \geq \hat{S}]$ is getting lower and therefore the probability that the agents will succeed in cleaning the contaminated area is increasing. We are showing that by choosing small enough \hat{S} so we know that the agents will succeed in cleaning the rest of the 'dirty' region, we will be able to guarantee with high probability the whole cleaning of the 'dirty' region. For example, choosing \hat{S} to be in $o(k)$ will assure that for $S_t \leq \hat{S} \leq c \cdot k$ for some small constant c , the rest of the contaminated region will be cleaned in at most c time steps by the k cleaning agents.

An illustration of the bound on the cleaning time, as presented in Theorem 2, is shown in Figure 2 and the



(a) The bound probability Q_t as a function of \hat{S} for various δ values



(b) The bound probability Q_t as a function of \hat{S} for various p values

FIG. 3: An illustration of the probability produced by the bound presented in Theorem 2. In (a) we can see the bound probability Q_t for initial region's size $S_0 = 20000$, spreading probability $p = 0.5$ and number of agents $k = 150$ for the following of values of $\delta \in [0.4, 0.3, 0.25, 0.2]$. In (b) we can see Q_t for $S_0 = 20000$, $\delta = 0.3$ and $k = 150$ for the following of values of $p \in [0.6, 0.5, 0.4, 0.3]$

corresponding bound probability in Figure 3. Notice that as \hat{S} and δ increase the cleaning time decreases.

B. Using The Bound

In Theorem 2 we presented a bound which guarantees that the contaminated region's size will not be smaller than some predefined size \hat{S} with the bound probability, Q_t . We suggest a method which make this bound useful when one willing to be guaranteed of successfully cleaning of the contaminated area with some desired probability Q_t in a

certain model, where the initial contaminated region's size is S_0 , each one of the tiles in the surrounding neighborhood of the contaminated area has a probability p to be contaminated by each one of its neighbors's contamination spreads and with k cleaning agents.

Notice that the only free variables left in the bound are the analysis parameters δ and \hat{S} . We should also notice the fact that as δ decreases the “usefulness” of the bound decreases (see Figure 3) because when δ is closer to 0, the model turn to the deterministic variant of the *cooperative cleaning* problem. In this variant of the problem the bound, as presented in Theorem 2, will “predict” that the contamination will spread exactly by the *potential boundary* mean in every step (as shown in V A). Furthermore, as δ increases, although the “usefulness” of the bound increases, the predicted bound is the naive one, where at each step there are no spreads and all the k agents clean perfectly - i.e. the size of the contaminated region at time step t will be exactly $S_t = S_0 - t \cdot k$ (which can be guaranteed in high probability).

We suggest the following method, in order to eliminate the need to identify the analysis parameters. Once someone willing to use this bound he should provide the desired bound probability - Q_t and the parameter of the model. Then for each value of δ in the range of $[0, 1]$ he should find the corresponding value of the minimal \hat{S} which satisfies the inequality $Pr[S_t \geq \hat{S}] \geq Q_t$ as illustrated in Figure 4. As $\delta \in \mathbf{R}$ - i.e. a real number, once using this method we should choose the granularity of δ for which calculate the appropriate \hat{S} .

Notice that there can be exists some minimal value of $\delta = \delta_{MIN}$, where for any value of $\delta < \delta_{MIN}$ there is no solution for the bound inequality. Also notice that there can be exists some maximal value of $\delta = \delta_{MAX}$, where for any value of $\delta > \delta_{MAX}$ the corresponding \hat{S} is the same as for δ_{MAX} .

Furthermore, for each pair of values of δ and \hat{S} there exists its corresponding cleaning process of k agents with initial region's size S_0 as demonstrated in Figure 4(b). Each curve bounds the cleaning process from S_0 to the applicable \hat{S} .

As we are interested in finding the tightest bound, looking at the frontier of the bounds, as shown in Figure 4(b), we can combine the relevant curves to one comprehensive bound. This bounds integrates the bound for a specific range of j values of $\delta \in [\delta_{i_1}, \delta_{i_2}, \dots, \delta_{i_j}]$, where for each time step t we choose the maximal S_t as illustrated in Figure 5(a).

Notice that the combined bound is independent of the selection of values for analysis parameters δ and \hat{S} . Also we can notice that this bound limits the contaminated region's size to some minimal \hat{S}_{MIN} where we almost certain that the agents will succeed in terminating the cleaning process successfully.

Notice that the inequality in Theorem 2 bounds the probability that the contaminated region's size at time step t will not be smaller than \hat{S} , therefore, an increase in the spreadings probability causes to an increase in the expected contaminated region's size and thus increases the probability $(\hat{q})^t$ (as demonstrated in Figure 3(b)).

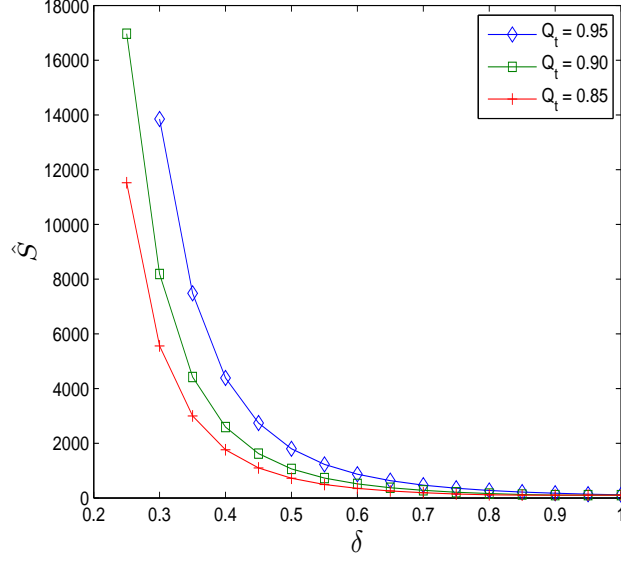
Notice that as illustrated in Figure 5(b), as the number of agents increases the probabilistic and the deterministic bounds are more similar. This result is not surprising considering the method we presented. In our method, in order to make the bound tighter, we favor the lower values of δ . As the number of agents increases the bound can be guaranteed in the desired probability with lower values of δ and as δ decreases the model becomes more similar to the deterministic one.

C. Parameters Selection

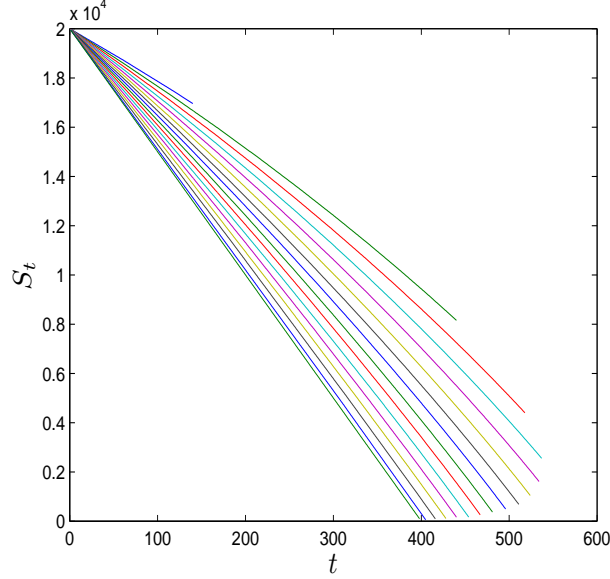
One of the problems of the bound as brought in Section V A is the nature of the probabilistic bounds to decay to 0 (as shown in Figure 3), which caused due to the fact that the bound probability Q_t is a product of each step's probability, q_t , and because as t increase Q_t decreases. One of the reasons which explains this problem is a bad selection of parameters - e.g. in the bound for the cleaning time (Eq. 4 a selection of too small \hat{S} will lead to fast decay in the probability. Furthermore, there exist trade-offs, when selecting the parameters' values, between the bound results and the the probability which guarantees its likelihood (e.g. see Figure 7).

One way to avoid this problem is by selecting \hat{S} as big as possible, as in Figures 3 and 7. As \hat{S} increases the probability of each time step, q_t , increases and so the total probability Q_t . Another technique for eliminating this problem is by “wrapping” number of time steps into one, thus artificially decreasing the time and therefore decreasing the power of q_t in Q_t (in Lemma 1).

Another example of the trade-off in choosing the parameters can be shown in Figure 3(b) where Q_t is illustrated for various values of the probability p . Interestingly, as p decreases our confidence in the bound result is decreasing although we know that the the agents performing the cleaning protocol have a better chance to successfully complete their work.



(a) \hat{S} as a function of δ for various bound probability Q_t values

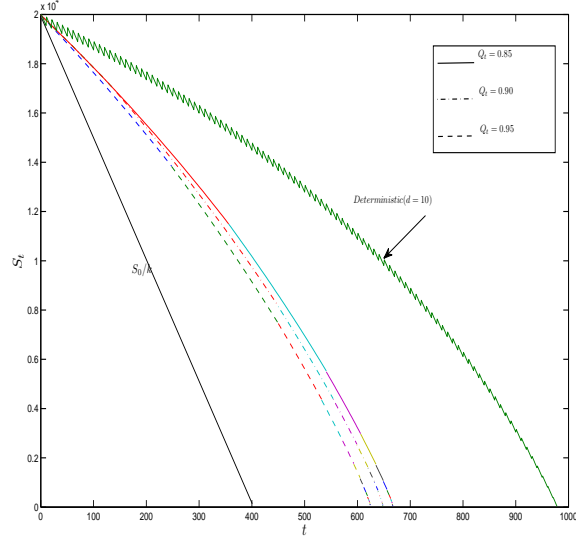


(b) The region's size S_t for various values of $\langle Q_t, \hat{S} \rangle$ pairs

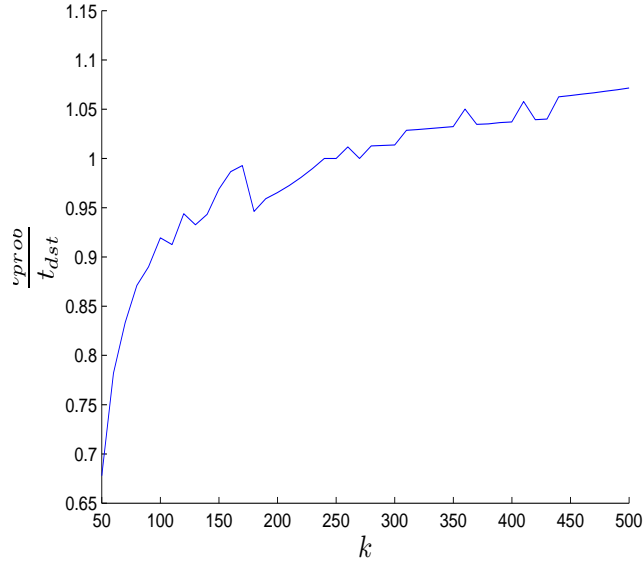
FIG. 4: Figure (a) is an illustration of \hat{S} as a function of δ using the bound presented in Theorem 2 for the following model parameters – $S_0 = 20000$, $p = 0.1$ and $k = 50$ for various values of Q_t . Figure (b) is an illustration of the cleaning process for various pairs of values of \hat{S} and δ as shown in Figure (a) for the same model parameters with $Q_t = 0.95$.

VI. IMPOSSIBILITY RESULT

While the theoretical lower bound presented in Section V can decrease the uncertainty whether a solution for the cleaning problem with certain number of agents exists, one might be interested in the opposite question, namely — how can we guarantee that a group of agents *will not* be able to successfully accomplish the cleaning work (regardless of the cleaning protocol being used or the contaminated region's properties e.g. its shape and spreading probability). A first impossibility result for the deterministic case of the Cooperative Cleaners problem was published in [24]), where an initial size that is impossible to clean (using any protocol) was demonstrated. In this paper, we extend this result in order to be applicable for stochastically expanding domains as well. Consequently, we will set the impossibility



(a) The combined bounds for various values of the bound probability Q_t



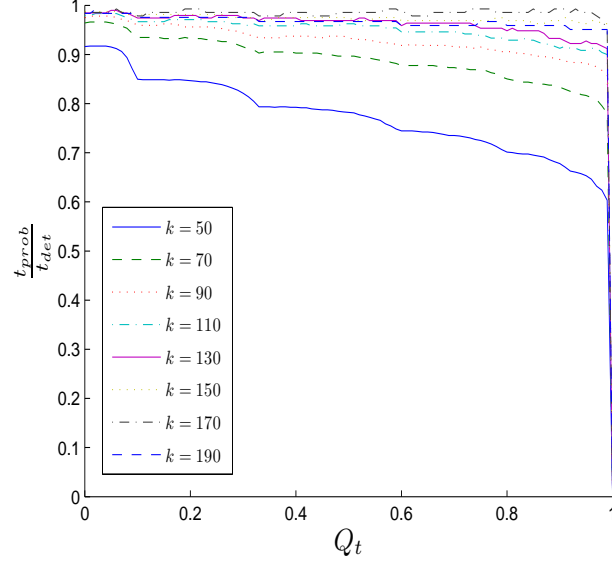
(b) The ratio between the bounds as function of the number of agents k

FIG. 5: Figure (a) An illustration of the combined bound for the bounds as shown in Figures 4(a) and 4(b) for the same model parameters – $S_0 = 20000$, $p = 0.1$ and $k = 50$ for various values of Q_t compared to the deterministic model with $d = 10$ and to the naive bound S_0/k . Figure (b) compare the deterministic bound and the probabilistic bound as a function of the desired guaranteeing probability - Q_t for various contaminated region's sizes and number of cleaning agents.

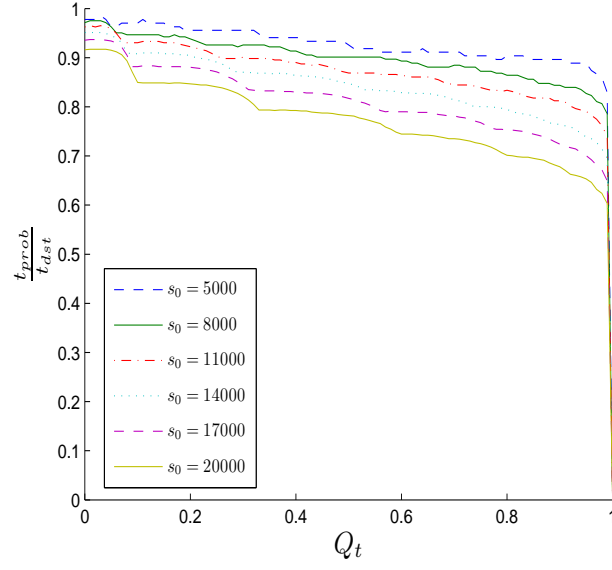
result with probabilistic restrictions as follows :

Theorem 3. *Using any cleaning protocol, k agents cleaning a contaminated region , where each tile in the potential boundary may be contaminate by already contaminated neighboring tiles with some probability p in each time step, will not be able to cleat this contaminated region if :*

$$S_0 > \left\lfloor \frac{k^2}{8 \cdot p^2} + k + \frac{1}{2} \right\rfloor$$



(a) The ratio between the bounds as function of the bound probability Q_t for various number of agents k



(b) The ratio between the bounds as function of the bound probability Q_t for various values of initial region's size S_0

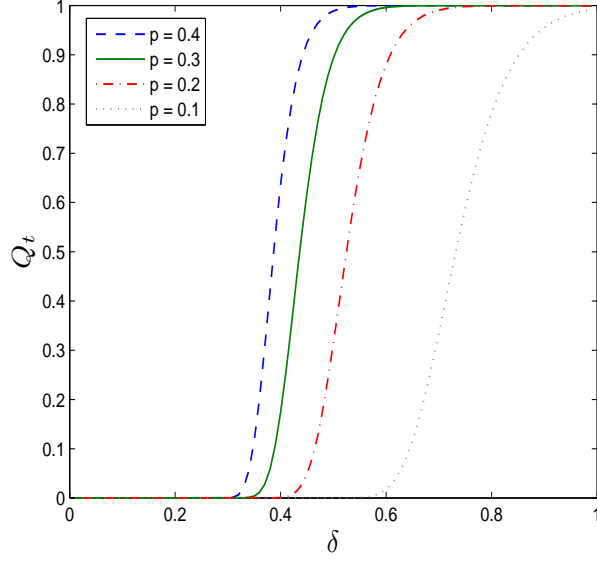
FIG. 6: Figures (b) and (a) compare the deterministic bound and the probabilistic bound as a function of the desired guaranteeing probability - Q_t for various contaminated region's sizes and number of cleaning agents.

with the probability Q_t for any time step t - i.e. :

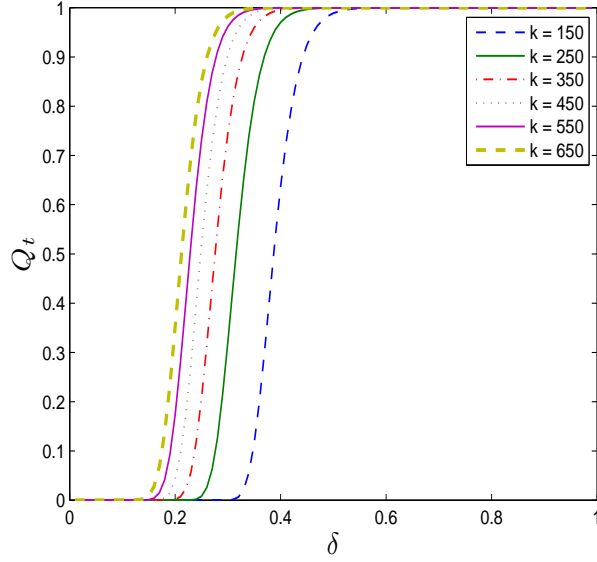
$$\forall t \Pr [S_t \geq S_0] = Q_t$$

Proof. Firstly, we shall require that the contaminated region's size increases between each time step, guaranteeing us that the contaminated region's size will keep on growing, and thus impossible to be cleaned. Therefore we want that in each time step t the size of the contaminated region will be bigger than the previous one - i.e. :

$$S_{t+1} - S_t > 0$$



(a) The bound probability Q_t as a function of δ for various p values



(b) The bound probability Q_t as a function of δ for various numbers of agents

FIG. 7: Parameters selection. In (a) we can see Q_t for $S_0 = 20000$ and $k = 150$ for the following of values of $p \in [0.4, 0.3, 0.2, 0.1]$. In (b) we can see Q_t for $S_0 = 20000$ and $p = 0.4$ for the following of values of $k \in [150, 250, 350, 450, 550, 650]$

Using Theorem 1 we know that :

$$S_{t+1} \geq S_t - k + \left\lfloor 2 \cdot (1 - \delta) p \cdot \sqrt{2 \cdot (S_t - k) - 1} \right\rfloor$$

and therefore we shell require that :

$$\left\lfloor 2 \cdot (1 - \delta) p \cdot \sqrt{2 \cdot (S_t - k) - 1} \right\rfloor - k > 0$$

Choosing, without loss of generality, $t = 0$ and after some arithmetics, we see that :

$$S_0 > \left\lfloor \frac{k^2}{8(1-\delta)^2 p^2} + k + \frac{1}{2} \right\rfloor$$

As S_0 as a function of δ is monotonically increasing and tends to infinity as δ tends to 1, we can lower bound S_0 with $\delta = 0$, therefore :

$$S_0 > \left\lfloor \frac{k^2}{8 \cdot p^2} + k + \frac{1}{2} \right\rfloor \quad (5)$$

We would like this process to continue for t time steps, thus applying the same method for all time steps and using Def. 10, we get that :

$$\forall t \Pr [S_t \geq S_0] = Q_t$$

□

□

Notice that Theorem 3 produces two results - the first one is the minimal initial region's size S_0 which guarantees that the cleaning agents *will not* be able to successfully accomplish the cleaning process and second one is the corresponding probability Q_t in which this S_0 can be guaranteed. Also notice that in order to evaluate Q_t , one should use the appropriate S_0 and δ .

Interestingly, the results demonstrated in Figure. 8 of the impossibility result as presented in Theorem 3, as the number of cleaning agents increases the probability which we can guarantee the minimal initial region's size S_0 also increases - in other words, although as the number of agents increases, the corresponding minimal initial region's size also increases and the probability in which we can guarantee that the agents will not be able to successfully clean the region increases as well.

VII. EXPERIMENTAL RESULTS

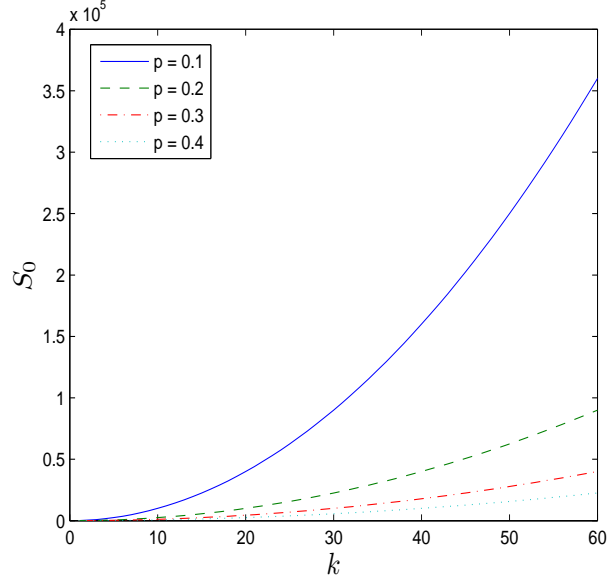
In previous work [2] a cleaning protocol for a group of K agents collaboratively cleaning an expending region was developed, called **SWEEP**. The performance of the algorithm was analyzed in [2], both analytically and experimentally. We had implemented the algorithm for a revised environment - where stochastic changes take place, as defined throughout this paper. We have conducted an extensive observational analysis of the performance of this algorithm. Exhaustive simulations were carried out, examining the cleaning activity of the protocol for various combinations of parameters - namely, number of agents, spreading probability (or spreading time in the deterministic model) and geometric features of the contaminated region. All the results were averaged over at least 1000 deferent runnings in order to get a statistical significance. In the deterministic model we average the results over the all the possible starting positions of the agents. Notice that, in order to minimize the running time, all running were stopped after some significant time and we consider these runnings as failure - i.e. these results are not included in the average calculation and not counted in the success percentage.

Some of the experimental results are presented in Figure 9 comparing the probabilistic model and the deterministic model over three deferent shapes (circle, square and cross) with range of number of agents. Notice the interesting phenomenon, where adding more agents may cause to an increase in the cleaning time due to the delay caused by the agents synchronization in the **SWEEP** protocol.

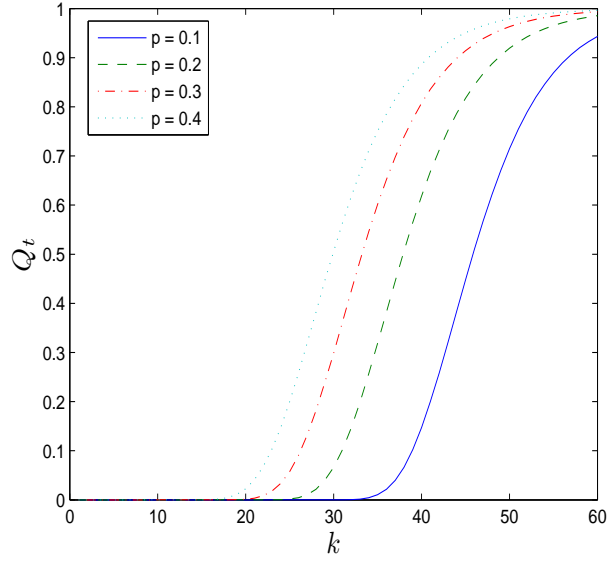
VIII. CONCLUSIONS

In this work we set the foundations of the stochastic model for the *Cooperative Cleaning* problem and introduce, for the first time, the basic definitions describe this problem. We present two lower bounds on the contaminated region's size and on the cleaning time under the limitation of this probabilistic model and demonstrate an impossibility result on the number of agents which are essential for a successful completion cleaning a contaminated region.

One of the ways these results could be further enhanced would involve analyzing the transition process between several possible "states" of the system, as a Markov process. Once analyzing the process as a *Markov's Chain*, we can get the stationary distribution of the process i.e. the probability to get to each one of the ending states (totally clean or impossible to clean).



(a) The minimal initial region's size S_0 as a function of the number of agents k for various probabilities.



(b) The guaranteed probability Q_t as a function of the number of agents k for various probabilities.

FIG. 8: An illustration of the impossibility result as presented in Theorem 3. In (a) we can see the minimal initial region's size S_0 for number of agents $k = [1..60]$ and spreading probability $p = [0.1, 0.2, 0.3, 0.4]$. In (b) we can see the corresponding probability Q_t .

It is also interesting to mention the similarity of this work to recent works done in the field of influence models in social networks. For example, in [38] the authors demonstrate that a probabilistic local rule can efficiently simulate the spread of ideas in a social network. Combining this result with our work can generate a unique approach for

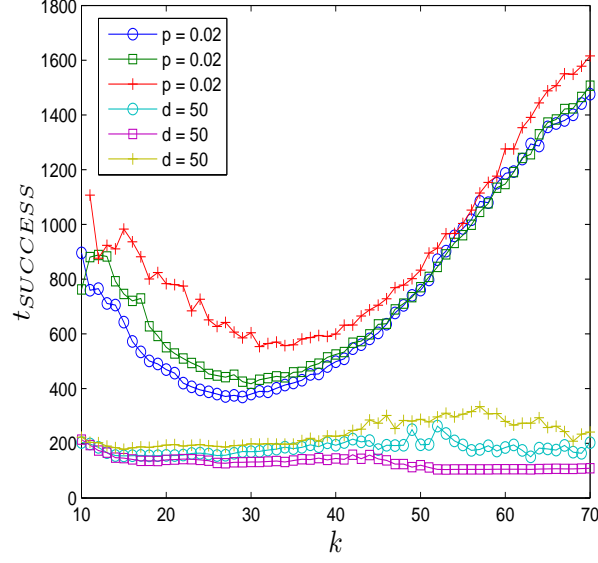
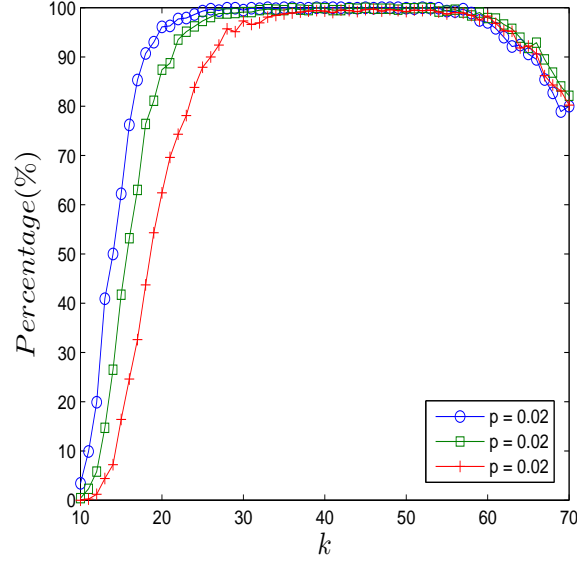
(a) Average $T_{SUCCESS}$ as a function of the number of agents k (b) Success percentage as a function of the number of agents k

FIG. 9: Experimental results for various number of agents for spheric and squared contaminated region with starting size $S_0 = 500$ and with spreading probability of $p = 0.02$ (notice that all the running were stopped after 3000 time steps). In (a) can see the results compared to the deterministic model results (with $d = \frac{1}{p}$ and in (b) the success percentage

analyzing dynamics of information flow in social networks.

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